An Alternative Definition of “Natural Number” in Frege’s Concept Script

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In *The Foundations of Arithmetic,* Gottlob Frege sets out on the ambitious task of demonstrating that the justification of all mathematical facts rests solely on the validity of logic. This project, now termed logicism, although ultimately failing as a result of various paradoxes found in naive set theory, was of incredible mathematical and philosophical import. For this project, Frege created Concept Script, the first quantificational predicate logic, which is considered by George and Velleman to be “the greatest contribution to the development of the axiomatic system since the ancient Greeks” (15).

Frege starts by reducing to logic what he sees as the most fundamental mathematical entity of all: the number. In doing this, he analyzes language in such a novel way that he is considered by many to be the father of analytic philosophy. In this essay, I give an overview of Frege’s definition of natural number and also present an alternative one. I show that by assuming the axiom of choice, Peano’s postulates can be proven using this alternative definition and that it has the advantage of not being impredicative.

Frege’s Definition of “Number”

Frege arrives at his definition of a number through multiple stages, moving from each stage after showing that it is unsatisfactory in some way. Frege insists that his definition should give numbers the status of “self-subsistent objects” and allow us to recognize a number when we come across it again (Frege 68). Ultimately we are given a definition that says that

> *the Number which belongs to the concept F is the extension of the concept “equal to the concept F” (Frege 79).*

We say the concept F is equal to G, or equinumerous as George and Velleman have translated it, if there exists a one-to-one correspondence between the objects in the extension of F onto the objects in the extension of G. Under this definition, for example, the number 3 would be the class of all sets that have exactly three elements. With the above definition, we can now say that "n is a *cardinal number* if and only if there exists a concept F such that $n =$ the number of F" (George and Velleman 32).

Frege now turns his attention on defining the number 0, the number 1, and what it means for one number to succeed (follow directly) another:

> 0 *is the Number which belongs to the concept "not identical with itself" (Frege 87).*
>
> 1 *is the Number which belongs to the concept "identical with* 0*" (Frege 90).*
>
> *"there exists a concept* F*, and an object falling under it* x*, such that the Number which belongs to the concept* F *is* n *and the Number which belongs to the concept 'falling under* F *but not identical with* x*' is* m*" is to mean the same as "*n *follows in the series of natural numbers directly after* m*" (Frege 89).*

From these definitions, Frege can deduce some simple facts about the numbers 0 and 1, such as the fact that 1 succeeds 0. He continues in this fashion and shows how to construct a successor of a natural number n. Frege says that "the Number which belongs to the concept 'member of the series of natural numbers ending with n' follows in the series of natural numbers directly after n" (Frege 94). If we refer to this number as $n + 1$, we can restate this as

$n + 1 =$ the number of the concept "member of the set $\{0, 1, \dots, n\}$".

This is a very ingenious definition, as it creates a concept with an extension containing $n + 1$ elements by using the $n + 1$ numbers defined before it. Interestingly, the extensions of the concepts used for this purpose correspond to the Von Neumann ordinals. For example, while the Von Neumann ordinal 2 is $\{0, 1\}$, for Frege, the number 2 is the number of the concept "member of the set $\{0, 1\}$".

Frege's Definition of "Natural Number"

Frege finishes off his exposition with a definition of a natural or finite number, which is stated by George and Velleman as:

> *k is a natural number if and only if*
> $\forall F[(F0 \wedge \forall x \forall y((Fx \wedge Sxy) \to Fy)) \to Fk]$ *(35).*

Note that Sxy means y succeeds x. It is important to note that this definition is impredicative. It relies on the fact that F will quantify over the range of all possible concepts, including the concept *natural number*. We rely on this fact to eliminate objects that are not natural numbers but that are still elements of inductive sets.

An Alternative Definition of "Natural Number"

I now present an alternative definition for natural number where

> k is a natural number if and only if $k = 0$ or there exists a concept F such that k is the number of F and there exists an x falling under F such that k is not the number of the concept "falls under F but is not identical to x".

This definition has a lot of similarities with the definition of successor, and as such we can restate it on those terms. In other words, k is a natural number if and only if it is 0 or if it is not equal to its predecessor. If we assume the axiom of choice then every infinite set has a countable subset, and it is easy to find a bijection between a given infinite set, and that same set missing a single element. As a result, no infinite set is a natural number according to our definition, since all infinite sets are equal to their predecessors.

This definition is not impredicative as it does not rely on quantifying over the range of all inductive sets, but instead gives a way of recognizing an object as a natural number through a property that natural numbers alone possess.

I will now show that this definition of natural number can be used to prove Peano's postulates, given by George and Velleman as:

> *(i) 0 is a natural number*
> *(ii) For any x and y, if x is a natural number and y succeeds x, then y is a natural number.*
> *(iii) Every natural number has a unique successor.*
> *(iv) 0 is not the successor of any natural number.*
> *(v) If x and y are natural numbers and $x \neq y$, then the successor of x is not equal to the successor of y.*
> *(vi) For any F, if $F0$ and F is hereditary with respect to successor and k is any natural number, then Fk (37).*

That (i) is satisfied is obvious given the alternative definition. The proof of (ii) is as follows:

Consider x and y where x is a natural number and y succeeds x. Since y succeeds x we know there exists a concept F where $y =$ the number of F and that there exists a z falling under F such that $x =$ the number of the concept "falls under F but is not identical to z", which we will call G. Assume for contradiction that y is not a natural number. From the alternative definition of natural number, this implies that $y = x$ and that the number of F = the number of G. However, this implies that x = the number of F = the number of G, meaning that x is equal to its predecessor, contradicting the fact that x is a natural number. Therefore y is also a natural number.

The proof of the uniqueness of a successor, when it exists, does not depend on the definition of natural number used and is given by George and Velleman on page 33. Therefore to prove postulate (iii), we need only show the existence of a successor. With Frege's definition of natural number, this proof relies on the principle of mathematical induction, whereas with the alternative definition a more direct proof can be given:

Consider k a natural number. We know that there exists a concept F such that $k =$ the number of F. It is clear that there are not an infinite number of objects falling under F, for if this were the case then k would be equal to its predecessor. Therefore, assuming that there exist an infinite number of logical objects, there exists an x that does not fall under F, and we can consider the concept $G =$ "falls under F or is identical to x".

Since F is the same as the concept "falls under G but is not identical to x", we can see that the number of G succeeds k, as required.

The proofs of postulates (iv) and (v) do not involve the definition of natural number, but instead use Frege's construction of ever-larger natural numbers and the definition of successor. Lastly, postulate (vi), also known as the principle of mathematical induction, can be proven like so:

Consider k a natural number and a concept F such that $F0$ and F is hereditary with respect to successor.

If $k = 0$ then k falls under F and we are finished. Consider $k > 0$. We know that there exists a concept G such that k is the number of G. From the reasoning given in our proof to postulate (iii), we know that there are only a finite number of objects that fall under G. We can label these objects $x_1, x_2, \ldots, x_n$. We can construct the predecessor of k as the number of the concept "member of the set $\{x_1, \ldots, x_{n-1}\}$".

Continuing in this manner, we can construct a chain of predecessors, and after $n - 1$ steps, we will have arrived at the number of the concept "member of the set $\{x_1\}$". After one final step, we will have arrived at the number of the concept "member of the set $\varnothing$", which is the number 0. We have shown that 0 is an ancestor of k, and since F is hereditary with respect to successor, k falls under F.

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References

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